Cavitation Flow Past Airfoils

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TULIN and Geurst (1,2)[1] investigated the plane problem of a flow past cavitating airfoils under assumptions usually made in problems of cavitation streams: 1) the fluid is weightless and incompressible; 2) the boundaries of the cavity are the flow lines at which the flow velocity is constant. In addition, the following limitations are superposed on the airfoils and cavity: the angles of attack are small; the airfoils are infinitely thin; the origin of the cavity coincides with the sharp leading edge of the airfoil; and the ratio of cavity thickness to length is small.

We will consider below an analogous problem for airfoils of an arbitrary shape, including airfoils with a rounded leading edge.

1 Let us assume that the ratio of cavity thickness to its length is small compared with unity.

We will replace the cavity by a system of sources continuously distributed along the contour of the airfoil (Fig. 1) between points A and C and in the wake along axis x between points C and B (r is the cavity boundary).

We will use the auxiliary plane of the complex variable $\zeta = \xi + i\eta$ whose region outside a disk of unit radius is conformably transformed by a certain function $z = f(\zeta)$ to region $z = x + iy$ outside the contour of the airfoil. The correspondence of the boundaries in planes z and ζ is shown in Fig. 1.

In plane ζ the sources are located on the arc of disk A_1C_1 and on line C_1B_1. In order to satisfy the conditions of the impenetrability of the contour of the disk (wing) we must place an additional system of sources within it by known rules.

Using the assumptions of the linear theory of flow past thin bodies, we can establish the following relations between the intensity of the sources and the shape of the cavity:

$$q(\theta) = 2v_1 \left|\frac{dz}{d\zeta}\right| \frac{dr}{d\theta} \qquad q(\xi) = 2v_1 \left|\frac{dz}{d\zeta}\right| \frac{d\eta}{d\xi} \qquad [1.1]$$

Here $q(\theta)$ and $q(\xi)$ are the intensities of sources distributed on the arc of the disk θ and axis ξ, respectively; v_1 is the value of flow velocity at the cavity boundary in the physical plane; $dr/d\theta$ is the tangent of the slope angle between tangents to the cavity and disk; $d\eta/d\xi$ is the tangent of the slope angle between the tangent to the cavity contour and axis ξ.

The velocity v_1 is related with the cavitation number by the relation

$$\varkappa = \frac{{v_1}^2}{{v_\infty}^2} - 1 \qquad [1.2]$$

where v_∞ is the value of the velocity of the incident flow at infinity.

Equating the expressions for the velocity at the cavity boundary in the plane of the disk and for the velocity of the flow past the contour of the disk in the presence of a system of sources, we derive the integral equation for determining the intensity of the latter:

$$v_1 \left|\frac{dz}{d\zeta}\right| = v \qquad [1.3]$$

where

$$v = 2v_\infty \sin(\theta - \alpha) - \frac{\Gamma}{2\pi} - \frac{1}{4\pi}\int_0^{\theta_2} \frac{q(\varphi)\sin(\theta - \varphi)d\varphi}{1 - \cos(\theta - \varphi)} - \frac{\sin\theta}{2\pi}\int_1^{\xi_1} \frac{\xi' q(\xi')d\xi'}{\xi'^2 - 2\xi'\cos\theta + 1} \qquad [1.4]$$

for sources distributed within the limits of the contour of the disk, and

$$v = v_\infty \cos\alpha\left(1 - \frac{1}{\xi^2}\right) + \frac{\xi^2 - 1}{4\pi\xi}\int_0^{\theta_2} \frac{q(\varphi)d\varphi}{\xi^2 - 2\xi\cos\varphi + 1} + \frac{1}{4\pi}\int_1^{\xi_1} \frac{q(\xi')d\xi'}{\xi - \xi'} + \frac{1}{4\pi\xi}\int_1^{\xi_1} \frac{q(\xi')d\xi'}{\xi\xi' - 1} \qquad [1.5]$$

for sources distributed along axis ξ.

Here α is the angle of attack of the airfoil calculated from the zero-lift angle; Γ is the circulation around the cavitating airfoil; θ_2 is the angular coordinate of the point where the cavity separates from the disk contour; and ξ_1 is the abscissa of the point where the cavity closes.

Circulation around the cavitating airfoil can be determined from the Chaplygin-Doukowski condition for a plane flow past the trailing edge

$$\frac{\Gamma}{2\pi} = -2v_\infty \sin\alpha + \frac{1}{4\pi}\int_0^{\theta_2} \frac{q(\varphi)\sin\varphi d\varphi}{1 - \cos\varphi} \qquad [1.6]$$

If the cavity is closed on the surface of the airfoil, the right-hand side of Eq. [1.3] is simplified

$$v = 2v_\infty \sin(\theta - \alpha) - \frac{\Gamma}{2\pi} - \frac{1}{4\pi}\int_{\theta_1}^{\theta_2} \frac{q(\varphi)\sin(\theta - \varphi)d\varphi}{1 - \cos(\theta - \varphi)} \qquad [1.7]$$

where θ_1 is the angular coordinate of the closing point of the cavity on the contour of the disk.

Two additional conditions are needed to determine the previously unknown coordinates of the points of separation and closing of the cavity. As the first condition, we will use that of the closed nature of the cavity, designating the zero equality of the total intensity of the sources

$$\int_{\xi_1}^{1} q(\xi) + \int_{\theta}^{\theta_2} q(\varphi)d\varphi = 0 \qquad [1.8]$$

As the second condition, we will take that of the tangency of the cavity boundary to the contour of the airfoil at the separation point of the cavity

$$\frac{dq}{d\theta} = 0 \quad \text{when} \quad \theta = \theta_2 \qquad [1.9]$$

Condition [1.9] is not fulfilled when the cavity separates at a fixed point of the airfoil.

2 Let us consider the case of the cavity closing on the airfoil contour which is described by integral equation [1.3] with the right-hand side expressed by Eq. [1.7]. For all values of $(\varphi - \theta)^2 < 4\pi^2$ the kernel of the integral equation can be represented by the following Laurent series:

$$\frac{\sin(\varphi - \theta)}{1 - \cos(\varphi - \theta)} - \frac{\sin\varphi}{1 - \cos\varphi} = \frac{2}{\varphi - \theta} - \frac{2}{\varphi} - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!}\left[\left(\frac{\varphi - \theta}{2}\right)^{2k-1} - \frac{\varphi^{2k-1}}{2}\right] \qquad [2.1]$$

Translated from Izvestiia Akademii Nauk SSSR, Otdel. Tekh. Nauk, Mekhanika i Mashinostroenie (Bulletin of the Academy of Sciences USSR, Div. Tech. Sci., Mechanics and Machine Construction), no. 6, 117–120 (1960). Translated by Primary Sources, New York.

[1] Numbers in parentheses indicate References at end of paper.

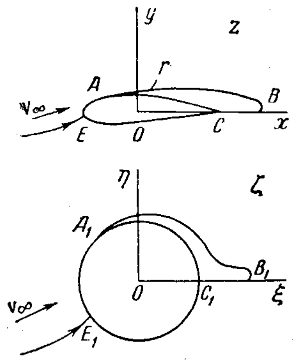

Fig. 1

where $|B_{2k}|$ are the absolute values of Bernoulli numbers. We will transform the regular part of the series

$$\sum_{i=1}^{\infty} \theta^{2i} \sum_{k=1+i}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} \binom{2k-1}{2i} \varphi^{2(k-1)-1} - \sum_{i=1}^{\infty} \theta^{2i-1} \sum_{k=i}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} \binom{2k-1}{2i-1} \varphi^{2(k-i)}$$

where $\binom{2k-1}{2i}$ is the number of combinations of $2k - 1$ elements taken $2i$ at a time.

Introducing the designations

$$A_1 = \frac{1}{2\pi v_\infty} \int_{\theta_1}^{\theta_2} \frac{q d\varphi}{\varphi}$$

$$a_{k,2i} = \int_{\theta_1}^{\theta_2} \varphi^{2(k-i)-1} q d\varphi \qquad [2.2]$$

$$a_{k,2i-1} = \int_{\theta_1}^{\theta_2} \varphi^{2(k-i)} q d\varphi$$

we will represent the integral equation in the following form:

$$v_1 \left|\frac{dz}{d\zeta}\right| - 2v_\infty \sin(\theta - \alpha) - 2v_\infty \sin\alpha + A_1 v_\infty + \sum_{i=1}^{\infty} \theta^{2i} \sum_{k=1+i}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} \binom{2k-1}{2i} a_{k,2i} + \sum_{i=1}^{\infty} \theta^{2i-1} \times \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} \binom{2k-1}{2i-1} a_{k,2i-1} = \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} \frac{q d\varphi}{\varphi - \theta} \qquad [2.3]$$

We will use one of the known forms of treating the indicated Eq. [3] corresponding to the problem considered

$$q(\theta) = -\frac{2}{\pi} \sqrt{\frac{\theta_2 - \theta}{\theta - \theta_1}} \int_{\theta_1}^{\theta_2} \sqrt{\frac{\varphi - \theta_1}{\theta_2 - \varphi}} \frac{f_1(\varphi) d\varphi}{\varphi - \theta} \qquad [2.4]$$

where the left-hand part of integral Eq. [2.3] is designated through $f_1(\varphi)$. The unknown constant coefficients A_1, $a_{k,2i}$ and $a_{k,2i-1}$ entering [2.4] can be determined from an infinite system of algebraic equations based on Eqs. [2.2]. As a consequence of the very rapid decrease in the coefficients of unknowns, we can restrict ourselves to a finite, small number of terms in the indicated system. It is sufficient to retain only the first term in the regular part of series [2.1] for the range of variations in the difference $\varphi - \theta$ from 0 to π which is of practical interest. The approximate values of the kernel K^0 (dashed line in Fig. 2) differs very little from the exact values of kernel K (solid curve):

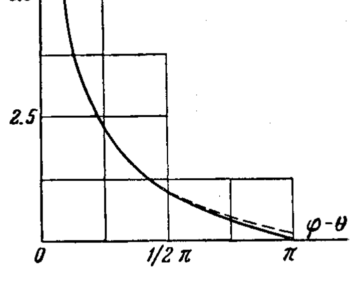

Fig. 2

$$K^0 = \frac{2}{\varphi - \theta} - \frac{\varphi - \theta}{6} \qquad K = \frac{\sin[(\varphi - \theta)]}{1 - \cos(\varphi - \theta)}$$

Treatment of the integral equation in this case can be written in the following form:

$$q(\theta') = \frac{2v_\infty}{\pi} \sqrt{\frac{1 - \theta'}{1 + \theta'}} \int_{-1}^{+1} \left[A - 2\sin\alpha - 2\sin\left(\frac{\theta_2 - \theta_1}{2}\varphi' + \frac{\theta_2 + \theta_1}{2} - \alpha\right) + \sqrt{1 + \varkappa}\left|\frac{dz}{d\zeta}\right|\right] \sqrt{\frac{1 + \varphi'}{1 - \varphi'}} \frac{d\varphi'}{\theta' - \varphi'} \qquad [2.5]$$

Here

$$A = \frac{1}{2\pi v_\infty} \int_{-1}^{+1} \frac{q(\varphi') d\varphi'}{\varphi' + m} \qquad m = \frac{\theta_2 + \theta_1}{\theta_2 - \theta_1}$$

$$\varphi = \frac{\theta_2 - \theta_1}{2} \varphi' + \frac{\theta_2 + \theta_1}{2} \qquad [2.6]$$

$$\theta = \frac{\theta_2 - \theta_1}{2} \theta' + \frac{\theta_2 + \theta_1}{2}$$

Using Eq. [1.6] and the Joukowski formula for the lift of an airfoil, we can determine the lift coefficient of a cavitating airfoil C_y:

$$C_y = C_{y0}\left(1 - \frac{A - \Delta A}{2 \sin\alpha}\right)$$

$$\Delta A = \frac{1}{24\pi} \left(\frac{\theta_2 - \theta_1}{2}\right)^2 \int_{-1}^{+1} (\varphi' + m) q(\varphi') d\varphi' \qquad [2.7]$$

Here C_{y0} is the lift coefficient for a noncavitating airfoil.

The resistance and moment coefficients can be found by direct integration with respect to the airfoil surface of dimensionless pressures that are related to velocities on the airfoil surface by the relation

$$\bar{p} = 1 - \left(\frac{v}{v_\infty}\right)^2$$

$$\frac{v}{v_\infty} = \frac{2\sin(\theta - \alpha) + 2\sin\alpha - A + J}{|dz/d\zeta|} \qquad [2.8]$$

$$J = \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} \frac{\sin(\varphi - \theta)}{1 - \cos(\varphi - \theta)} q(\varphi) d\zeta \qquad [2.9]$$

Using Eq. [1.1] we can obtain the formula for determining the distances of the points of the cavity boundary to the contour of the airfoil along the normal to the latter

$$y = \frac{\theta_1 - \theta_2}{4\pi v_1} \int_1^{\theta'} q(\varphi') d\varphi' \qquad [2.10]$$

As an illustration, Figs. 3–5 show the results of calculations for a flat plate. Fig. 3 shows curves characterizing the change in the ratio of cavity length to chord $1/b$ as a function of parameter $\varkappa^{-1} \sin\alpha$. The curve referring to $\alpha \to 0$ coincides with the analogous curve in Ref. 2, according to which the relation between $\varkappa$ and α will be linear. Fig. 4 shows for three fixed values of $1/b$ the shape of the cavity boundary:

$$Y = \frac{\sqrt{1 + \varkappa}}{b \sin\alpha} y$$

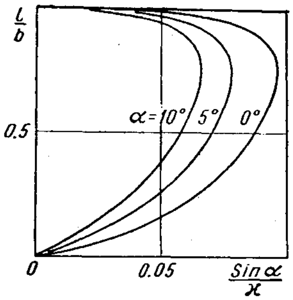

Fig. 3

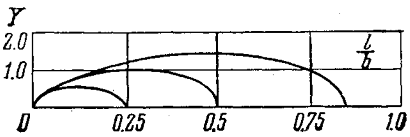

Fig. 4

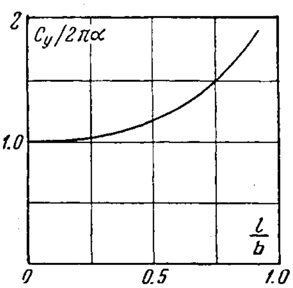

Fig. 5

Fig. 5 shows the ratio of the lift coefficients for cavitating and unseparated flows as a function of the relative cavity length.

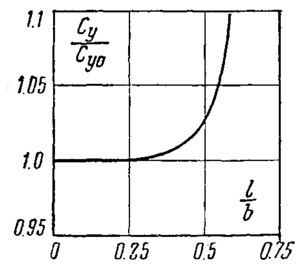

Fig. 6

Fig. 6 shows the results of the calculations for arcs of a disk for the case when the cavity separates from the convex side of the airfoil. This figure depicts the dependence of the ratio of the lift forces of the airfoil for cavitating and unseparated flows on the relative length of the cavity at angles of attack corresponding to shockless flow past the leading edge of the cavitating airfoil. As the calculations indicate, as the cavity length increases, the point of its separation from the airfoil does not shift noticeably but remains at a distance equal to approximately 0.413 of the chord from the leading edge.

—*Submitted March 16, 1960*

References

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Reviewer's Comment

This paper presents an interesting theoretical analysis of cavitating flow for flow over an airfoil at an angle of attack in a uniform stream. The analysis is based on two-dimensional potential flow theory used in conjunction with the theory of functions of a complex variable.

The most important aspect of this paper is the treatment of flow about arbitrary airfoil configurations but at small angles of attack. It is thus a generalization of those studies dealing only with flat plates or arc-type airfoils. The type of cavity examined is assumed to be long in comparison with its width, but is closed on the airfoil contour, as depicted in the author's Fig. 1. Continuous source distributions are employed to generate the flow configurations. Ultimately the problem involves solution of an integral equation. An approximate solution of this equation is obtained.

This paper considers flow over a flat plate with separation as an illustration of the technique. It is interesting to note that this particular problem recently has received attention by two United States research men. As such, their publications, Refs. 1 and 2, are an interesting supplement to the present paper. Ref. 1 treats the problem of an inclined flat plate at an arbitrary angle of attack. An extension of the work reported in Ref. 3 is employed, namely, an analysis of free streamline flow based on the flow hodograph. Results obtained agree very well with experiment, and the mathematical analysis is straightforward and uncomplicated. However, the paper does not treat closed cavities. Ref. 2 approaches the problem of cavitating flow over a flat plate in a manner almost identical to that presented in Ref. 1. In addition to the analysis of a fully developed cavitation region, the closed cavitation problem is also treated and hence is somewhat similar to the subject paper in that respect. Again theory and experiment show excellent agreement. The beauty of both Refs. 1 and 2 is the comparative simplicity of the analysis. The present paper suffers somewhat from this lack of simplicity, but has the added value of significant generalization.

Additional research in this general area is presented in the reference list.

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